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On long waves with cross-wind in an atmosphere

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Asymptotic equations of long waves in a compressible medium of infinite height with arbitrary density and wind profiles, including cross-wind, are systematically derived to provide the mathematical description of certain pressure-front phenomena observed in the atmosphere. It is assumed that the final state of the wave is a plane wave moving with nearly constant velocity, even though its velocity field has a component transverse to the direction of propagation. The coefficients of the asymptotic long-wave equations are found to depend on the equilibrium profiles of the density and of the velocity component in the direction of propagation of the wave, but not on the profile of the transverse velocity. Any horizontal direction of propagation is found to be possible, even for waves of permanent type.

1. Introduction

Meteorological stations in the midwestern United States have recorded pressure variations which show a sudden rise over a short time (Tepper 1954), and their importance is underlined by the observation that severe local storms often follow the sweep of the pressure jumps. It is in fact thought (Tepper 1954; Abdullah 1956) that such jumps may act as the excitation for the formation of tornadoes. The most common genesis of the reported pressure disturbances appears to be due to a cold front invading a region with an inversion layer.

Pressure jump lines, constructed according to isochrone patterns, indicate that both oscillatory and one-shot branches occur which are long gravity waves in the general category of the so-called undular jumps of hydraulics. The explanation of such phenomena is beyond the scope of the linear theory of small-amplitude gravity waves. On the other hand, application of the long-wave theory of surface waves on water is impeded by two major difficulties, namely the compressibility of the atmosphere and the need to take account of cross-wind, since the motion in the atmosphere is in general three-dimensional. A substantially new approach is therefore needed, which extends the available two-dimensional long-wave theory to a stratified compressible medium of infinite height with three-dimensional shear. Such a theory will be presented here, and it can be applied in turn, with only minor modifications, to long waves in general water bodies of constant depth with density stratification and cross-currents, in order to explain internal undular fronts in the ocean as observed by Frassetto (1964).

The mathematical model developed below concerns a compressible medium of infinite height with arbitrary, continuous density and wind profiles in the initial steady state. Effects of viscosity, condensation and the earth's rotation and curvature are neglected. It is assumed that an unsteady wave has been generated by some disturbance, and that it has formed rectilinear 'crests' and 'troughs' which propagate in the direction normal to their own with a velocity close to some fixed speed c. It is also assumed that the relevant horizontal scale L of the wave (which cannot always be associated with a wavelength) is large compared with the height H of the homogeneous atmosphere, and that the wave is a near-steady motion for an observer travelling with the velocity c in the direction of propagation. Hydraulic long waves of such a type have been studied by many authors since Airy (1845) and Boussinesq (1871). Recently permanent waves in incompressible fluids of more geophysical interest have been considered by Peters & Stoker (1960), Benjamin (1962, 1966), Long (1965), Peters (1966*a*, *b*), and Benney (1966), in whose paper oblique solitary waves were first mentioned. It has been known that their propagation velocity must be close to some 'critical' velocity of the linear theory of small-amplitude gravity waves (e.g. Keller 1948; Peters & Stoker 1960); the same is true in the atmosphere of our problem.

The stretching transformation used here to develop asymptotic equations for the long waves is an extension of a transformation due to Gardner & Morikawa (1960) and Meyer (1967). It will be formulated in §2, and the zeroth and first approximations to the perturbation field will be obtained in §3 in terms of a single unknown function, which will then be discussed in §4. It is an intrinsic feature of long-wave theory that different types of waves are possible, and their separate discovery in hydraulics led to the long-wave 'paradox', first elucidated by Ursell (1953). To avoid a similar confusion in meteorology, a unified derivation of the asymptotic equations is presented here, which indicates clearly the respective areas of application of the different asymptotic equations (§4). Accordingly, most of the discussion of the results is best postponed to that section.

Results common to all wave types include the following. Propagation is possible in any horizontal direction in the atmosphere, regardless of the vertical distributions (or, briefly, profiles) of the wind components in the equilibrium state. The critical velocity (§3) is the same for all wave types and depends only on the equilibrium profiles of the density and of the velocity component in the direction of propagation. The horizontal distribution of the first-order perturbation density, pressure, and velocity is similarly independent of equilibrium cross-wind; but the vertical profile of the perturbation velocity (§3) depends on both components of the equilibrium wind, and a three-dimensional shear in the equilibrium atmosphere implies a three-dimensional perturbation velocity. Finally, the model to be presented does not aim at a realistic description of the initial disturbance causing the wave, but it will emerge in §4 that there are grounds for the conjecture that the model can describe a large part of the process by which the wave gradually develops into its final state.

The results of this study may seem depressing to the meteorologist, because they indicate the possibility of so many different long-wave motions in the atmosphere. On the other hand, a complete observational exploration of any given disturbance, at all atmospheric levels and over a time interval, is practically unattainable. Even to diagnose the type of an atmospheric disturbance under study may therefore require the use of whatever clues the theory can offer. Once the type has been diagnosed, moreover, the theory will be seen to be capable of deducing a complete picture of the disturbance from only a rough knowledge of the equilibrium atmosphere and quite scanty observational data on the disturbance.

2. Formulation

We consider a compressible medium of infinite height supported by a rigid plane bottom surface, and assume that the pressure of the medium tends to zero as the height tends to infinity. A co-ordinate system (x, y, z) is chosen such that the (x, y)-plane coincides with the bottom surface and z is positive upward. Initially the medium is in a state of equilibrium, and the velocity vector field $(u_0(z), v_0(z), 0)$ and the density distribution $\rho_0(z) > 0$ for $0 \leqslant z < \infty$ are given. It

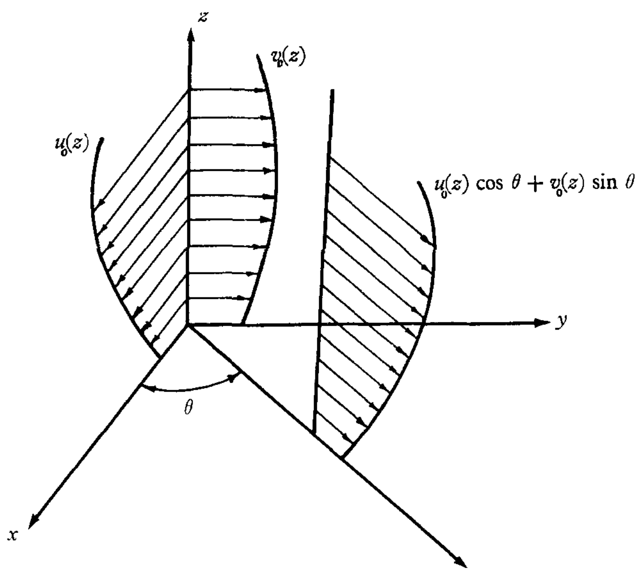

FIGURE 1. Wind profiles and the direction of wave propagation.

is assumed that an unsteady wave travels into the medium, and, as time t increases, the unsteady wave approaches as its final state a plane wave which appears steady, or at least near-steady, to an observer travelling with a suitable constant speed $c > 0$ in the direction $(\cos\theta, \sin\theta, 0)$, where θ is the angle between the direction of propagation and the x-axis (figure 1).

The medium is assumed to be compressible and inviscid, so that its motion is governed by the equations

$$\partial\rho/\partial t + \operatorname{div}(\rho q) = 0, \tag{1}$$

$$\rho\,\partial q/\partial t + \rho(q\,.\,\mathrm{grad})\,q = -\mathrm{grad}\,p - (0, 0, \rho q), \tag{2}$$

$$p/p_b = P(\rho/\rho_b), \tag{3}$$

subject to the conditions

$$p \to 0 \quad \text{as} \quad z \to \infty, \tag{4}$$

and

$$w^* = 0 \quad \text{at} \quad z = 0. \tag{5}$$

Here $q = (u, v, w^*)$ is the velocity vector, p the pressure, ρ the density, g the gravitational constant, $p_b > 0$ and $\rho_b > 0$ are respectively the pressure and density at $z = 0$ in the state of equilibrium, and $P(\rho/\rho_b)$ is any function of ρ/ρ_b defined in $0 < \rho < \infty$ such that

$$dP/d\rho > 0 \quad \text{for} \quad \rho > 0 \quad \text{and} \quad P \to 0 \quad \text{as} \quad \rho \to 0.$$

We assume for definiteness that ρ_0 and its derivatives with respect to z tend to zero exponentially as $z \to \infty$ and that all the functions concerned are continuously differentiable as many times as required and, together with their derivatives, remain bounded as $z \to \infty$.

To make the governing equations non-dimensional, it is convenient to measure x, y, z in units of the height $H = p_b/g\rho_b$ of the 'homogeneous' atmosphere, t in units of H/c, u, v and w^* in units of c, p in units of $\rho_b c^2$, and ρ in units of ρ_b. It is known that the propagation speed of long water waves must be close to a critical speed near which linear theory indicates steady motions to be unstable (Peters & Stoker 1960). Accordingly, we suppose that the propagation speed c is close to a critical speed $(gH/l)^{\frac{1}{2}}$, to be determined in §3, and follow Friedrichs & Hyers (1954) in regarding the difference between the two speeds as a small parameter. To express this non-dimensionally, write $c^2 = gH/\lambda$ and

$$l - \lambda = \epsilon^k \quad (k \geqslant 1).$$

Non-degenerate solutions will emerge only if the variables x and y are stretched to reflect the fact that the waves are of horizontal scale large compared with the height H of the homogeneous atmosphere. Since four different stretching laws will turn out to be of physical interest, a general stretching transformation is introduced by

$$\xi = \epsilon^{\frac{1}{2}m}(ax + by - t), \quad \tau = \epsilon^{\frac{1}{2}m+n}t, \quad w^* = \epsilon^{\frac{1}{2}m}w, \quad (m, n \geqslant 1),$$

where $a = \cos\theta$, $b = \sin\theta$. Thus ξ measures distance from the wave front, on an appropriate scale, and the stretching parameter in τ reflects the approach to steadiness in the proper Galilean frame.

If we now consider u, v, w, p and ρ as functions of ξ, τ and z, equations (1) to (5) become (with $U = au + bv - 1$)

$$\epsilon^n \partial\rho/\partial\tau + \partial(\rho U)/\partial\xi + \partial(\rho w)/\partial z = 0, \tag{6}$$

$$\epsilon^n \rho\, \partial u/\partial\tau + \rho U\, \partial u/\partial\xi + \rho w\, \partial u/\partial z = -a\, \partial p/\partial\xi, \tag{7}$$

$$\epsilon^n \rho\, \partial v/\partial\tau + \rho U\, \partial v/\partial\xi + \rho w\, \partial v/\partial z = -b\, \partial p/\partial\xi, \tag{8}$$

$$\epsilon^{m+n} \rho\, \partial w/\partial\tau + \epsilon^m \rho U\, \partial w/\partial\xi + \epsilon^m w\, \partial w/\partial z = -\partial p/\partial z + \epsilon^k \rho - l\rho, \tag{9}$$

$$p = -\epsilon^k P(\rho) + lP(\rho), \tag{10}$$

$$p(\xi, \tau, \infty) = 0, \quad w(\xi, \tau, 0) = 0. \tag{11}$$

Assume now that the dependent variables have asymptotic expansions of the form

$$\phi = \phi_0(z) + \epsilon^k \phi_1(\xi, \tau, z) + \epsilon^{k+1} \phi_2(\xi, \tau, z) + \ldots, \tag{12}$$

where ϕ stands for any one of U, u, v, w, p and ρ, and that ϕ tends to the initial steady state as $\xi \to \infty$. Substitution of (12) in (6) to (11) then yields a sequence of equations and conditions for successive approximations depending on the values of k, m and n. We shall only consider the cases (I) $k = m = n = 1$, (II) $k = m = 1$, $n > 1$, (III) $k = n = 1, m > 1$, and (IV) $m = n = 1, k > 1$, which yield physically interesting results; for all of them, the zeroth and first approximations have the same form.

3. Zeroth and first approximations

The equations for the zeroth approximation govern the equilibrium state; with $w_0 \equiv 0$ and u_0, v_0, ρ_0 given as functions only of z, they serve only to determine the zeroth approximation for the pressure as

$$p_0 = lP(\rho_0) = l\int_z^\infty \rho_0(z')\,dz'. \tag{13}$$

The equations for the first approximation are

$$U_0 \partial\rho_1/\partial\xi + \rho_0 \partial U_1/\partial\xi + \partial(\rho_0 w_1)/\partial z = 0, \tag{14}$$

$$\rho_0 U_0 \partial u_1/\partial\xi + \rho_0 w_1 du_0/dz = -a\,\partial p_1/\partial\xi, \tag{15}$$

$$\rho_0 U_0 \partial v_1/\partial\xi + \rho_0 w_1 dv_0/dz = -b\,\partial p_1/\partial\xi, \tag{16}$$

$$-\partial p_1/\partial z + \rho_0 - l\rho_1 = 0, \tag{17}$$

$$p_1 = \rho_1 dp_0/d\rho_0 - l^{-1}\rho_0, \tag{18}$$

subject to the conditions

$$p_1(\xi, \tau, \infty) = 0, \quad w_1(\xi, \tau, 0) = 0. \tag{19}$$

From (17) to (19) and (13),

$$p_1 = -l^{-1}p_0 + \rho_0 f, \tag{20}$$

$$\rho_1 = -l^{-1} f d\rho_0/dz, \tag{21}$$

where $f = f(\xi, \tau)$, and, on the assumption that w_1 remains bounded as $z \to \infty$, it follows from (14) to (16), after an integration with respect to z, that

$$w_1 = U_0 G(z)\,\partial f/\partial\xi, \tag{22}$$

$$u_1 = -(G du_0/dz + aU_0^{-1})f, \tag{23}$$

$$v_1 = -(G dv_0/dz + bU_0^{-1})f, \tag{24}$$

where
$$G(z) = l^{-1} - \rho_0^{-1}\int_z^\infty \rho_0(z')\,[U_0(z')]^{-2}\,dz',$$

for a wave advancing into an atmosphere in equilibrium, where the boundary condition $\rho \to \rho_0$, $u \to u_0$, $v \to v_0$ as $\xi \to \infty$ implies $f \to 0$ as $\xi \to \infty$, by (21).

It is now seen from (23), (24) that a wave of the assumed, general type can exist only if U_0 has no zero. This means that the propagation velocity c of long waves must differ from the values at all levels in the atmosphere of the component of wind velocity in the propagation direction. In fact, the critical speed (and zeroth approximation to the propagation speed), $(gH/l)^{\frac{1}{2}}$, is given by

$$l^{-1} = \int_0^\infty \rho_0(z)\,[U_0 z)]^{-2}\,dz, \tag{25}$$

because the condition $w_1 = 0$ at $z = 0$ implies $G(0) = 0$, by (22).

Since $G(z)$ is now also determined, (20) to (24) express all the first-order deviations from equilibrium in terms of the single unknown function $f(\xi, \tau)$. This function describes their variation with time and with distance from the wave front. Their variation with height is already given explicitly by (20) to (24). The vertical distribution of the velocity perturbation is seen to depend rather intricately on the equilibrium wind profile; by contrast, the vertical distribution of the pressure and density perturbations depend on that profile only through the critical-speed parameter l. If we use dimensional variables in (25), then

$$l = \lambda + \epsilon = \left[\int_0^\infty \rho_0(z)\,(au_0(z)+bv_0(z)-c)^{-2}dz\right]^{-1} \rho_b/c^2,$$

and, to the zeroth approximation, c satisfies

$$\left[\int_0^\infty \rho_0(z)\,(au_0(z)+bv_0(z)-c)^{-2}dz\right]^{-1} = p_b/\rho_b^2.$$

Following an argument by Peters (1966*a*) we can show that if $au_0(z)+bv_0(z)$ is a non-negative continuous function for $0 \leqslant z < \infty$ and m and M are respectively its minimum and maximum then there exist two roots of c, say c_-, c_+ with $c_- < m$ and $c_+ > M$. A discussion of the physical aspects of the values of c for the incompressible case was given by Benjamin (1962). c_+ is the propagation velocity for waves travelling in the sense of the equilibrium longitudinal velocity, relative to the fluid, and c_- is the propagation velocity for waves travelling in the opposite sense. In case c_- is negative, then a wave propagating in the opposite direction of θ may take place.

4. The equations for f

To determine f, we must proceed to the equations for the second approximation.

Case I. $k = m = n = 1$

This corresponds to a wave for which the smallest scale L of the horizontal variations is large compared with the height H of the homogeneous atmosphere, for which the amplitude (in units of H) is small of order $(H/L)^2$, and for which the unsteadiness (for the proper Galilean observer) has become so small that its time scale T is large compared with $H/c \simeq (lH/g)^{\frac{1}{2}}$ in such a way that $L^3(Hl/g)^{\frac{1}{2}}/(H^3T)$ is of the order of unity. (6) to (12) then give

$$U_0\,\partial\rho_2/\partial\xi + \rho_0\,\partial U_2/\partial\xi + \partial(\rho_0 w_2)/\partial z = -\partial\rho_1/\partial\tau - \partial(\rho_1 U_1)/\partial\xi - \partial(\rho_1 w_1)/\partial z = G_1, \quad (26)$$

$$\rho_0 U_0\,\partial U_2/\partial\xi + \partial p_2/\partial\xi + \rho_0 w_2\,dU_0/dz = -\rho_0\,\partial U_1/\partial\tau - \rho_0 U_1\,\partial U_1/\partial\xi$$
$$-\rho_1 U_0\,\partial U_1/\partial\xi - \rho_1 w_1\,dU_0/dz - \rho_0 w_1\,\partial U_1/\partial z = G_2, \quad (27)$$

$$\rho_0 U_0\,\partial w_1/\partial\xi = -\partial p_2/\partial z + \rho_1 - l\rho_2, \quad (28)$$

$$p_2 = \rho_2\,dp_0/d\rho_0 + (\tfrac{1}{2})\rho_1^2\,d^2p_2/d\rho_0^2 - l^{-1}\rho_1\,dp_0/d\rho_0, \quad (29)$$

subject to the conditions

$$p_2(\xi,\tau,\infty) = 0, \quad w_2(\xi,\tau,0) = 0. \quad (30)$$

(13), (21), (28) and (29) imply

$$p_2 = (-2l)^{-1} f^2 d\rho_0/dz - \rho_0 \int_0^z U_0(\partial w_1/\partial \xi)\, dz' + \rho_0 f_1(\xi, \tau), \tag{31}$$

$$\rho_2 = (2l^2)^{-1} f^2 d^2\rho_0/dz^2 + l^{-1}(d\rho_0/dz) \int_0^z U_0(\partial w_1/\partial \xi)\, dz' - l^{-2}(f + l f_1)\, d\rho_0/dz, \tag{32}$$

where both p_2 and $\rho_2 \to 0$ as $z \to \infty$, since ρ_0 and $d\rho_0/dz \to 0$ exponentially by assumption, and it now follows from (26), (27) that

$$(d\rho_0/dz + l\rho_0 U_0^{-2})\, \partial f_1/\partial \xi - l\, \partial(\rho_0 w_2 U_0^{-2})/\partial z = G_3 \tag{33}$$

with

$$G_3 = l^{-1}(d^2\rho_0/dz^2) f \partial f/\partial z + \partial\left(\rho_0 \int_0^z U_0(\partial^2 w_1/\partial \xi^2)\, dz'\right)\Big/ \partial z + \partial \rho_1/\partial \xi - U_0 \rho_0 \partial^2 w_1/\partial \xi^2$$
$$+ U_0^{-2}(d\rho_0/dz) f \partial f/\partial \xi + l U_0^{-2} \rho_0 \int_0^z U_0(\partial^2 w_1/\partial \xi^2)\, dz' - l U_0^{-2} G_1 + l U_0^{-2} G_2.$$

Integration of (33) with respect to z gives w_2, and the condition that the connexion term w_2 must remain bounded implies

$$\int_0^\infty G_3\, dz = 0,$$

by (25) and (30). This consistency condition for the second approximation determines f. By some lengthy calculations, we finally obtain

$$m_0 \partial^3 f/\partial \xi^3 + m_1 f \partial f/\partial \xi + m_2 \partial f/\partial \xi + m_3 \partial f/\partial \tau = 0, \tag{34}$$

where

$$m_0 = -l \int_0^\infty \rho_0 U_0^2 G^2\, dz, \tag{35}$$

$$m_1 = -3l \left[\int_0^\infty \rho_0 U_0^{-4}\, dz + (\tfrac{1}{3}) l^{-2} (d\rho_0/dz)_{z=0}\right], \tag{36}$$

$$m_2 = l^{-1}, \tag{37}$$

$$m_3 = 2l \int_0^\infty \rho_0 U_0^{-3}\, dz. \tag{38}$$

The counterpart of (34) in water wave theory is known as the Korteweg–DeVries equation (Korteweg & DeVries 1895), a generalization of the so-called Boussinesq equation of solitary-wave theory. No analytic solution methods are known yet for (34), except when $\partial f/\partial \tau = 0$ or $m_1 = 0$, but one particular solution decaying like $t^{-\frac{2}{3}}$ has been identified (Gardner & Morikawa 1960) and some successful computations have been reported by Morton (1962), which show that (34) can explain the development of meteorological pressure jumps.

Case II. $k = m = 1,\ n > 1$

This corresponds to a wave of even less unsteadiness than in case I, so that the horizontal scale L is large compared with the height H of the homogeneous atmosphere, but the time scale T is so much larger than $(lH/g)^{\frac{1}{2}}$ that $L^3(Hl/g)^{\frac{1}{2}}/(H^3T)$ is small. The amplitude is again of order $(H/L)^2$. The equations for the second approximation then differ from (26) to (30) only by the absence of all the terms containing the derivative with respect to τ. The argument used for case I now leads to the Boussinesq equation

$$m_0 \partial^3 f/\partial \xi^3 + m_1 f \partial f/\partial \xi + m_2 \partial f/\partial \xi = 0, \tag{39}$$

and this integrates to

$$m_0(\partial f/\partial \xi)^2 = \alpha_2 + \alpha_1 f - m_2 f^2 - \tfrac{1}{3} m_1 f^3,$$

where α_1, α_2 may depend on τ. The present derivation shows that (39) indicates the possibility of three slightly different physical phenomena.

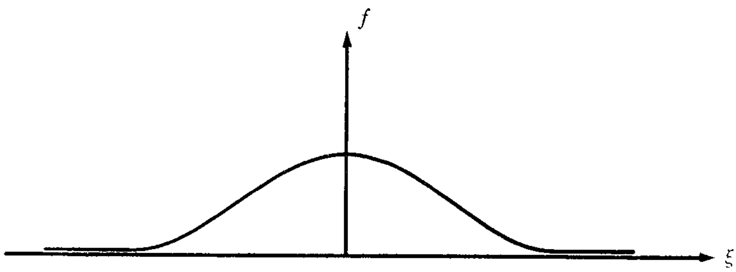

FIGURE 2. Solitary-wave profile.

First, (39) indicates the existence of solutions of (1) to (5) which are altogether independent of τ so that they are entirely steady in the proper Galilean frame and are therefore waves of 'permanent type'. Their amplitude, and the excess of their propagation speed over the critical speed, are small of the order of $(H/L)^2$, and they correspond to the 'solitary' and 'cnoidal' waves of hydraulic theory. For the solitary wave, $\alpha_1 = \alpha_2 = 0$, and the solution of (39) is

$$f = -(3m_2/m_1)\operatorname{sech}^2[(\xi/2)(-m_2/m_0)^{\frac{1}{2}}]$$

(figure 2). From (22) to (24), the first-order perturbation velocity components (in units of ϵc) are

$$U_1 = -(G dU_0/dz + U_0^{-1}) f,$$
$$V_1 = (b\, du_0/dz - a\, dv_0/dz) Gf,$$
$$w_1^* = \epsilon^{\frac{1}{2}} U_0 Gf,$$

where U_1 is the component in the propagation direction (or, briefly, longitudinal component) and V_1 the transverse, horizontal component; the vertical component, $w^* = \epsilon^{\frac{1}{2}} w_1$, is weak even compared with the horizontal components. The horizontal variation of all three components is described by the common factor f (figure 2), and accordingly these are one-shot, aperiodic waves which, once generated, can propagate without change of shape over long distances. It is seen from (25) and (35) to (38) that the horizontal variations depend only on the profiles of the density and of the longitudinal velocity component in the equilibrium atmosphere. On the other hand, the analysis shows that such internal waves may propagate in any horizontal direction, irrespective of the direction of the equilibrium wind, and that their perturbation velocity field is definitely three-dimensional, apart from very exceptional cases for which $V_1 \equiv 0$. For one such case, $a = 1$, $b = 0$, $v_0 = 0$, (39) reduces to the two-dimensional internal wave of Shen (1966). The two-dimensional solitary wave in a stratified liquid with shear has been discussed by Benjamin (1962, 1966).

Quite analogous remarks apply to the permanent cnoidal waves (for which α_1, α_2 are non-zero constants), except that these are periodic waves (Korteweg & DeVries 1895). They are distinct from linear, small-amplitude gravity waves in that they are not monochromatic and yet propagate without change of shape.

A second physical phenomenon covered by (39) is that of waves which are permanent—three-dimensional, solitary or cnoidal waves—to the first approximation, but show residual unsteadiness in higher approximations. How many approximations are steady depends on the magnitude of the exponent n, i.e. on how small $L^3(Hl/g)^{\frac{1}{2}}/(H^3T)$ is by comparison even with H/L.

For cnoidal waves finally, the possibility of waves arises for which the amplitude or wavelength or both change slowly with time, even to the first approximation, because α_1 and α_2 depend on τ. These coefficients can be interpreted in terms of the perturbation mass-flux, momentum-flux and energy-flux (Benjamin & Lighthill 1954) and their variation with time depends directly on the boundary conditions for the wave trains.

Case III. $k = n = 1$, $m > 1$

This corresponds to very long waves for which H/L is small compared with the square root of the amplitude. The equations for the second-order approximation are then again (26) to (30), except that the term containing $\partial w_1/\partial\xi$ vanishes in (28). Following the argument used in case I, we find

$$m_1 f\partial f/\partial\xi + m_2\partial f/\partial\xi + m_3\partial f/\partial\tau = 0,$$

whence $$df/d\tau = 0 \quad \text{when} \quad d\xi/d\tau = m_3^{-1}(m_2 + m_1 f),$$

if $m_3 \neq 0$. Thus any value of f propagates with a velocity dependent on f, and wave propagation without change of shape is not possible. The counterpart of this result in two-dimensional hydraulic theory is often called Airy theory.

This case is useful for the explanation of very long atmospheric waves in which a steepening of gradients during propagation is observed. For definiteness, consider only waves propagating in the sense of the equilibrium longitudinal velocity, relative to the fluid. Then $|\partial f/\partial\xi|$ increases with time (at constant f); i.e. the wave steepens wherever $m_1 f/m_3$ is a decreasing function of ξ (at fixed τ). Since $m_3 < 0$ for this sense of propagation, by (38), it is seen from the last equation that $|\partial f/\partial\xi|$ increases with time (at constant f); i.e. the wave steepens, wherever $m_1 f$ is an increasing function of ξ (at fixed τ). However, (36) shows that m_1 can, in principle, take either sign; it depends in a rather intricate manner on the equilibrium shear and density profiles, and both signs appear possible in realistic cases, even if there is no shear (Shen 1966). Such steepening will continue according to the last formula, until it has reduced the local horizontal scale to the order of magnitude approximate to case I. Indeed, cases III, I and II, in that order, form a very plausible sequence for explaining the gradual development of a wave of permanent type from an equilibrium atmosphere disturbed only by a very gentle horizontal gradient of pressure. This notion is confirmed, moreover, by a numerical solution of (34) in the context of two-dimensional surface waves on water (Peregrine 1966).

Case IV. $m = n = 1$, $k > 1$

This corresponds to long waves for which the horizontal scale L and the time scale are related as for case 1, but the amplitude is small compared with $(H/L)^2$, and the propagation speed is closer to the critical speed. From (6) to (12), the equations for the second-order approximation are found to be

$$U_0 \partial\rho_2/\partial\xi + \rho_0 \partial U_2/\partial\xi + \partial(\rho_0 w_2)/\partial z = -\partial\rho_1/\partial\tau,$$
$$\rho_0 U_0 \partial U_2/\partial\xi + \partial p_2/\partial\xi + \rho_0 w_2 dU_0/dz = -\rho_0 \partial U_1/\partial\tau,$$
$$\rho_0 U_0 \partial w_1/\partial\xi = -\partial p_2/\partial z - l\rho_2.$$
$$p_2 = \rho_2 dp_0/d\rho_0.$$

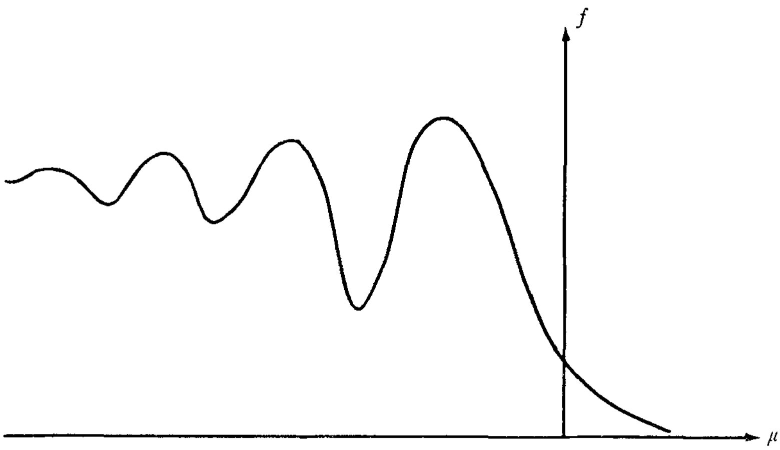

FIGURE 3. Dispersive-wave profile.

subject to the conditions

$$p_2(\xi,\tau,\infty) = 0, \quad w_2(\xi,\tau,0) = 0;$$

they lead to the conclusion that f satisfies the linear equation

$$m_0 \partial^3 f/\partial\xi^3 + m_3 \partial f/\partial\tau = 0, \tag{40}$$

where m_0 and m_3 are given by (35) and (38).

It is readily verified, if $m_0/m_3 > 0$, that (40), the boundary condition $f(\xi,\tau)\to 0$ as $\xi\to\infty$, and the initial condition $f(\xi,0) = 1$ for $\xi < 0$ and $f(\xi,0) = 0$ for $\xi > 0$ are all satisfied by (Gardner & Morikawa 1960)

$$f(\xi,\tau) = \int_\mu^\infty Ai(\mu')d\mu', \quad \mu = (3m_0\tau/m_3)^{-\frac{1}{3}}\xi,$$

(figure 3), where $Ai(\mu')$ denotes the Airy function (Jeffreys & Jeffreys 1946). This solution therefore describes the development in time of a small, but initially rather sharp change in the equilibrium state of the atmosphere. The solution represents a gradually spreading wave which propagates into the equilibrium atmosphere, with the deviations from equilibrium rising to a first maximum and thence—through a sequence of undulations of algebraically decreasing amplitude and wavelength (figure 3)—to a second and slightly different equilibrium state of the atmosphere.

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